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# Properties of Functions Generalized Convex with Respect to a WT-System

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Continuity and convergence properties of functions, generalized convex with respect to a continuous weak Tchebysheff system, are investigated. It is shown that, under certain non-degeneracy assumptions on the weak Tchebysheff system, every function in its generalized convex cone is continuous, and pointwise convergent sequences of generalized convex functions are uniformly convergent on compact subsets of the domain. Further, it is proved that, with respect to a continuous Tchebysheff system, $L_p$-convergence to a continuous function, pointwise convergence and uniform convergence of a sequence of generalized convex functions are equivalent on compact subsets of the domain.

## INTRODUCTION

A set $\{u_0,..., u_{n-1}\}$ of real-valued functions defined on a real interval is called a *weak Tchebysheff* (*WT-*) *system* if it is linearly independent and for all points $x_0 < \cdots < x_{n-1}$,

$$U\begin{pmatrix} 0,..., n-1 \\ x_0,..., x_{n-1} \end{pmatrix} = \det \{u_i(x_j)\}_0^{n-1} \geqslant 0.$$

If $\{u_0,..., u_n\}$ is a WT-system, then no element in $\operatorname{sp}\{u_0,..., u_{n-1}\}$ has more than $n-1$ sign changes [8]. A WT-system is called a *Tchebysheff* (*T-*) *system* in case all of these determinants are strictly positive. A function $f$ is said to be *generalized convex* with respect to $\{u_0,..., u_{n-1}\}$ provided

$$U\begin{pmatrix} 0,..., n-1, f \\ x_0,..., x_n \end{pmatrix} = \begin{vmatrix} u_0(x_0) & \cdots & u_0(x_0) \\ \vdots & & \vdots \\ u_{n-1}(x_0) & \cdots & u_{n-1}(x_n) \\ f(x_0) & \cdots & f(x_n) \end{vmatrix} \geqslant 0$$

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for all points $x_0 < \cdots < x_n$. The set of all such functions, denoted $C(u_0,\dots,u_{n-1})$, is a convex cone and is closed under pointwise convergence. For a more thorough explanation of these and other terms employed in this paper the reader is referred to [2] and [8].

This notion of generalized convexity had its inception in the 1926 dissertation of Hopf [1] who observed that a function $f$ is convex in the classical sense if, in our notation, $f \in C(1, x)$. The book [3] of Roberts and Varberg contains a discussion of the history and various formulations of generalized convexity as well as a lucid presentation of the properties of classical convex functions.

We single out two of these properties for investigation in the present paper. The first, well-known feature of convex functions is their continuity in the interior of their domain. The second is the fact that a pointwise convergent sequence of convex functions converges uniformly on closed subintervals of the domain. In Section 2 we consider both of these properties for generalized convex functions and demonstrate their validity when $\{u_0,\dots,u_{n-1}\}$ is a WT-system satisfying certain non-degeneracy assumptions. In Section 1 we prove a boundedness result for generalized convex functions, which, in conjunction with a result of Shisha and Travis [5], demonstrates the equivalence of $L_p$, pointwise and uniform convergence of a sequence of generalized convex functions when $\{u_0,\dots,u_{n-1}\}$ is a continuous T-system.

## 1

We start with a boundedness result for generalized convex functions, on which the rest of the results of this section depend.

LEMMA 1.1. *Let $\{u_0,\dots,u_{n-1}\}$ be a WT-system on $(a, b)$ and let $F \subset C(u_0,\dots,u_{n-1})$. If $a < x_0 < \cdots < x_n < b$ is a set of points for which*

$$U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{i-1},x_{i+1},\dots,x_n \end{pmatrix} > 0 \qquad (i = 0,\dots,n)$$

*and such that $F$ is pointwise bounded on $\{x_0,\dots,x_n\}$, then there exist elements $v_0,\dots,v_n \in \operatorname{sp}\{u_0,\dots,u_{n-1}\}$ such that for each $1 \leqslant i \leqslant n$ there are integers $0 \leqslant k, l \leqslant n$ such that for all $f \in F$ $v_k(x) \leqslant f(x) \leqslant v_l(x)$ for $x \in [x_{i-1}, x_i]$.*

*Proof.* Choose points $x_{-1} \in (a, x_0)$ and $x_{n+1} \in (x_n, b)$. Denote $\alpha_j = \sup\{f(x_j): f \in F\}$ and $\beta_j = \inf\{f(x_j): f \in F\}$ $(j = 0,\dots,n)$. By assumption these numbers are finite. Define

$$\begin{aligned} \gamma_j &= \beta_j \quad \text{if} \quad n-j \text{ is odd} \\ \gamma_j &= \alpha_j \quad \text{if} \quad n-j \text{ is even} \end{aligned} \qquad (j = 0,\dots,n).$$

If $x \in [x_{i-1}, x_{i+1}]$ for some $0 \leqslant i \leqslant n$, then for every $f \in F$

$$
\begin{aligned}
0 &\leqslant U\begin{pmatrix} 0,\dots, n-1, f \\ x_0,\dots, x_{i-1}, x, x_{i+1},\dots, x_n \end{pmatrix} \\
&= (-1)^{n-i} f(x)\cdot U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{i-1}, x_{i+1}, x_n \end{pmatrix} \\
&\quad + \sum_{\substack{j=0 \\ j\neq i}}^{n} (-1)^{n-j} f(x_j)\cdot U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{i-1}, x, x_{i+1},\dots, x_{j-1}, x_{j+1},\dots, x_n \end{pmatrix}.
\end{aligned}
$$

Denote

$$
v_{ij}(x) = \frac{U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{i-1}, x, x_{i+1},\dots, x_{j-1}, x_{j+1},\dots, x_n \end{pmatrix}}{U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{i-1}, x_{i+1}, x_n \end{pmatrix}}
$$

$(i, j = 0,\dots, n;\ i \neq j)$. By assumption, each of these denominators is positive, hence if $n-i$ is even, the above inequality yields

$$
\begin{aligned}
f(x) &\geqslant \sum_{\substack{j=0 \\ j\neq i}}^{n} (-1)^{n-j-1} f(x_j)\, v_{ij}(x) \\
&\geqslant \sum_{\substack{j=0 \\ j\neq i}}^{n} (-1)^{n-j-1} \gamma_j v_{ij}(x)
\end{aligned}
$$

for $x \in [x_{i-1}, x_{i+1}]$. For such an $i$ we set

$$
v_i(x) = \sum_{\substack{j=0 \\ j\neq i}}^{n} (-1)^{n-j-1} \gamma_j v_{ij}(x).
$$

Similarly, for $n-i$ odd we get

$$
\begin{aligned}
f(x) &\leqslant \sum_{\substack{j=0 \\ j\neq i}}^{n} (-1)^{n-j} f(x_j)\, v_{ij}(x) \\
&\leqslant \sum_{\substack{j=0 \\ j\neq i}}^{n} (-1)^{n-j} \gamma_j v_{ij}(x)
\end{aligned}
$$

for $x \in [x_{i-1}, x_{i+1}]$, and set

$$v_i(x) = \sum_{\substack{j=0 \\ j \neq i}}^{n} (-1)^{n-j} \gamma_j v_{ij}(x).$$

For example, if $n$ is even then we see that

$$\begin{array}{lll} v_0(x) \leqslant f(x) \leqslant v_1(x) & \text{on} & [x_0, x_1], \\ v_2(x) \leqslant f(x) \leqslant v_1(x) & \text{on} & [x_1, x_2], \\ v_2(x) \leqslant f(x) \leqslant v_3(x) & \text{on} & [x_2, x_3], \\ \vdots \quad \vdots \quad \vdots & \vdots & \vdots \\ v_n(x) \leqslant f(x) \leqslant v_{n-1}(x) & \text{on} & [x_{n-1}, x_n]. \end{array}$$

In general, on the interval $[x_{i-1}, x_i]$ $(1 \leqslant i \leqslant n)$, $v_{i-1}(x) \leqslant f(x) \leqslant v_i(x)$ if $n-i$ is odd, and $v_i(x) \leqslant f(x) \leqslant v_{i-1}(x)$ if $n-i$ is even. ∎

THEOREM 1.1. *Let $\{u_0,..., u_{n-1}\}$ be a T-system of functions that are bounded on closed subintervals of $(a, b)$ and let $F \subset C(u_0,..., u_{n-1})$ be pointwise bounded. Then $F$ is uniformly bounded on closed subintervals of $(a, b)$.*

As a further application of Lemma 1.1 we show that the property of having finite $L_p$-norm on closed subsintervals of $(a, b)$ is "hereditary."

THEOREM 1.2. *If $\{u_0,..., u_{n-1}\}$ is a T-system of functions in $L_p(a, b)$, $0 < p < \infty$, and $f \in C(u_0,..., u_{n-1})$ then $f \in L_p(c, d)$ for all $[c, d] \subset (a, b)$. Moreover, if $\{f_k\}_0^\infty \subset C(u_0,..., u_{n-1})$ and $\{f_k\}$ converges pointwise to a function $f$ then $f_k \to f$ in the $L_p$-norm on $[c, d] \subset (a, b)$.*

*Proof.* We note first that if $u_0,..., u_{n-1}$ are measurable functions then so too is $f \in C(u_0,..., u_{n-1})$. Indeed if $U = \mathrm{sp}\{u_i\}_0^{n-1}$ contains constants, then for each constant $k$, it follows from the fact that $\{u_0,..., u_{n-1}, f\}$ is a WT-system that $f^{-1}([k, \infty))$ is the union of a finite number of intervals, and thus is measurable. In the general case, we know that $U$ contains a positive function [8, p. 25], say, $v$. Hence, as $\{u_0/v,..., u_{n-1}/v\}$ is a T-system whose linear span contains the constants, and $f/v$ is generalized convex with respect to this system, the foregoing shows that $f/v$, and thus $f$, is measurable. It is now an easy consequence of Lemma 1.1 that $f \in L_p(c, d)$ for all $[c, d] \subset (a, b)$. If $\{f_k\}_0^\infty \subset C(u_0,..., u_{n-1})$ converges pointwise to $f$ then by Lemma 1.1 and the Dominated Convergence Theorem $f_k \to f$ in the $L_p$-norm on every $[c, d] \subset (a, b)$. ∎

In Section 2 we will show that, in certain cases, a pointwise convergent sequence of generalized convex functions actually converges uniformly on closed subsets of the domain. For continuous T-systems this is also a consequence of the following results.

THEOREM 1.3 (O. Shisha, S. Travis [5]). *Let* $\{u_0, \ldots, u_{n-1}\}$ *be a continuous T-system on* $(a, b)$ *and let* $\{f_k\}_0^\infty \subset C(u_0, \ldots, u_{n-1})$ *be a sequence converging in the* $L_p$*-norm to a continuous function* $(0 < p < \infty)$. *Then the convergence is uniform on closed subintervals of* $(a, b)$.

COROLLARY 1.1. *If* $\{u_0, \ldots, u_{n-1}\}$ *is a continuous T-system on* $(a, b)$ *then the* $L_p$*-convergence of a sequence of generalized convex functions to a continuous function, its pointwise convergence and its uniform convergence are equivalent on closed subintervals of* $(a, b)$.

*Proof.* This is an immediate consequence of Theorem 1.2 and Theorem 1.3. ∎

## SECTION 2

In this section we consider functions that are generalized convex with respect to a WT-system. $u_0, \ldots, u_{n-1}$ will be linearly independent functions and $U$ will always denote $\operatorname{sp}\{u_0, \ldots, u_{n-1}\}$.

DEFINITION 2.1. $U$ is said to be *degenerate* on an interval $I$ if there exists a non-trivial $u \in U$ such that $u \equiv 0$ on $I$.

We record that $U$ is degenerate on $I$ iff

$$U\begin{pmatrix} 0, \ldots, n-1 \\ x_0, \ldots, x_{n-1} \end{pmatrix} = 0$$

for all $x_0 < \cdots < x_{n-1}$ in $I$. T-Spaces (spaces spanned by T-systems) and linear spaces of analytic functions, such as polynomials, are non-degenerate; that is, no such non-trivial elements exist in these spaces.

LEMMA 2.1. *Let* $u_0, \ldots, u_{n-1}$ *be linearly independent on* $(a, b)$ *and let* $\xi \in (a, b)$. *If* $U$ *is degenerate neither on* $(a, \xi)$ *nor on* $(\xi, b)$ *then there exist points* $a < x_0 < \cdots < x_{n-2} < \xi < x_{n-1} < b$ *such that*

$$U\begin{pmatrix} 0, \ldots, n-1 \\ x_0, \ldots, x_{n-1} \end{pmatrix} \neq 0.$$

*Proof.* Since $U$ is not degenerate on $(a, \xi)$, there are points $a < x_0 < \cdots < x_{n-2} < \hat{x} < \xi$ such that

$$U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{n-2}, \hat{x} \end{pmatrix} \neq 0$$

(see, e.g., [6, Lemma 4.5]). Define

$$u(x) = U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{n-2}, x \end{pmatrix},$$

then $u$ is a non-trivial element of $U$ since $u(\hat{x}) \neq 0$. Moreover, $u$ cannot vanish identically in $(\xi, b)$ for then $U$ would be degenerate on $(\xi, b)$. Hence there is a point $x_{n-1} \in (\xi, b)$ for which $u(x_{n-1}) \neq 0$, and $x_0,\dots, x_{n-1}$ are the desired points. ∎

DEFINITION 2.2. A *vanishing point* for a linear space $U$ is a point $\xi$ such that $u(\xi) = 0$ for all $u \in U$.

As will subsequently become apparent, there is a delicate relationship between degeneracy in WT-spaces and vanishing points. This connection has been successfully exploited by several authors, primarily Stockenberg [6] and Zalik [7]. Among other things, they show that a WT-space on $(a, b)$ that is non-degenerate and has no vanishing points is a T-space. In [9] the author demonstrates that if a WT-space is not degenerate on any interval of the form $(a, \xi)$ or $(\xi, b)$, then the property of having no vanishing points is equivalent to the existence of a positive element in the space.

The next two lemmas provide determinant characterizations for vanishing points.

LEMMA 2.2. *Let $u_0,\dots, u_{n-1}$ be defined on $[a, b]$ and linearly independent in $(a, b)$. Then $\xi \in [a, b]$ is a vanishing point for $U$ iff*

$$U\begin{pmatrix} 0,\dots, n-1 \\ \xi, x_1,\dots, x_{n-1} \end{pmatrix} = 0$$

*for all $x_1,\dots, x_{n-1} \in (a, b)$.*

*Proof.* If $\xi$ is a vanishing point for $U$ then the assertion is clear. To prove the converse, we note that (since $u_0,\dots, u_{n-1}$ are linearly independent on $(a, b)$) we may select points $a < x_0 < \cdots < x_{n-1} < b$ such that

$$U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{n-1} \end{pmatrix} \neq 0.$$

Define

$$v_i(x) = U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{i-1}, x, x_{i+1},\dots, x_{n-1} \end{pmatrix} \qquad (i = 0,\dots, n-1);$$

then $v_0,\dots, v_{n-1}$ are linearly independent, since

$$v_i(x_j) = \delta_{ij} \cdot U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{n-1} \end{pmatrix},$$

and hence are a basis for $U$. However, by assumption $v_i(\xi) = 0$ $(i = 0,\dots, n-1)$, so $\xi$ is a vanishing point for $U$. ∎

LEMMA 2.3. *Let $u_0,\dots, u_{n-1}$ be linearly independent in $(a, b)$ and let $\xi \in (a, b)$. If $U$ is degenerate neither on $(a, \xi)$ nor on $(\xi, b)$ then $\xi$ is a vanishing point for $U$ iff*

$$U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{n-3}, \xi, x_{n-1} \end{pmatrix} = 0$$

*for all $a < x_0 < \cdots < x_{n-3} < \xi < x_{n-1} < b$.*

*Proof.* If $\xi$ is a vanishing point the assertion is clear. To prove the converse, choose points $a < x_0 < \cdots < x_{n-2} < \xi < x_{n-1} < b$ such that

$$U\begin{pmatrix} 0,\dots, n-1 \\ x_0,\dots, x_{n-1} \end{pmatrix} \neq 0,$$

as guaranteed by Lemma 2.1, and then proceed as in Lemma 2.2. ∎

The next definition and the lemma which follows it appear in [4, p. 180].

DEFINITION 2.3. A sequence of functions $\{f_k\}_0^\infty$ is said to *converge continuously* to a function $f$ at the point $x$ if for every sequence $\{x_k\}_0^\infty$ converging to $x$, $\lim_{k\to\infty} f_k(x_k) = f(x)$.

LEMMA 2.4. *$\{f_k\}_0^\infty$ converges continuously in $(a, b)$ to a continuous function $f$ iff it converges uniformly to $f$ on every compact subset of $(a, b)$.*

LEMMA 2.5. *Let $\{u_0,\dots, u_{n-1}\}$ be a continuous WT-system on $(a, b)$ and let $\xi \in (a, b)$ be a vanishing point for $U$. If $U$ is degenerate neither on $(a, \xi)$ nor on $(\xi, b)$ then any pointwise bounded sequence $\{f_k\}_0^\infty \subset C(u_0,\dots, u_{n-1})$ converges continuously to $0$ at $\xi$. In particular, every $f \in C(u_0,\dots, u_{n-1})$ is continuous at $\xi$ and vanishes there.*

*Proof.* Let $\{y_k\}_0^\infty \subset (a,b)$ be a sequence of points converging to $\xi$. As $U$ is not degenerate on $(a,\xi)$, there exist points $a < x_0 < \cdots < x_{n-1} < \xi$ for which

$$U\begin{pmatrix} 0,\ldots, n-1 \\ x_0,\ldots, x_{n-1} \end{pmatrix} > 0.$$

Since $\xi$ is a vanishing point, for $k$ large enough we can write

$$0 \leqslant U\begin{pmatrix} 0,\ldots, n-1, f_k \\ x_0,\ldots, x_{n-1}, y_k \end{pmatrix} = f_k(y_k)\cdot U\begin{pmatrix} 0,\ldots, n-1 \\ x_0,\ldots, x_{n-1} \end{pmatrix} + o(1).$$

Hence $\underline{\lim}_{k\to\infty} f_k(y_k) \geqslant 0$. By Lemma 2.1 there exist points $a < t_0 < \cdots < t_{n-2} < \xi < t_{n-1} < b$ such that

$$U\begin{pmatrix} 0,\ldots, n-1 \\ t_0,\ldots, t_{n-1} \end{pmatrix} > 0.$$

Hence, by a similar computation $\overline{\lim}_{k\to\infty} f_k(y_k) \leqslant 0$. Thus, $\overline{\lim}_{t\to\infty} f_k(y_k) \leqslant 0 \leqslant \underline{\lim}_{k\to\infty} f_k(y_k)$ and so $\lim_{k\to\infty} f_k(y_k) = 0$. This proves that $\{f_k\}$ converges continuously to 0 at $\xi$. By letting $f_k \equiv f$ for all $k$, we see that each $f \in C(u_0,\ldots, u_{n-1})$ is continuous at $\xi$ and vanishes there. ∎

THEOREM 2.1. *Let $\{u_0,\ldots, u_{n-1}\}$ be a continuous WT-system on $(a,b)$ and assume that $U$ is not degenerate on any interval of the form $(a,x)$ or $(x,b)$. If $\{f_k\}_0^\infty \subset C(u_0,\ldots, u_{n-1})$ converges pointwise in $(a,b)$ to a function $f$ then $\{f_k\}_0^\infty$ converges continuously to $f$ in $(a,b)$, and uniformly to $f$ on every compact subset of $(a,b)$. Moreover, every element of $C(u_0,\ldots, u_{n-1})$ is continuous in $(a,b)$.*

*Proof.* By Lemma 2.4 it suffices to show that $\{f_k\}$ converges continuously to $f$ at each point $x \in (a,b)$. Moreover, by Lemma 2.5 we may assume that $x$ is not a vanishing point. Let $\{y_k\}_0^\infty$ be a sequence in $(a,b)$ converging to $x$. Without loss of generality we will assume that $y_k \uparrow x$. Since $U$ is not degenerate on $(a,x)$, it follows from Lemma 2.2 that there are points $a < x_0 < \cdots < x_{n-2} < x$ for which

$$U\begin{pmatrix} 0,\ldots, n-1 \\ x_0,\ldots, x_{n-2}, x \end{pmatrix} > 0.$$

Let $k$ be large enough so that $a < x_0 < \cdots < x_{n-2} < y_k < x$. Then

$$0 \leqslant U\begin{pmatrix} 0,\dots,n-1,f_k \\ x_0,\dots,x_{n-2},y_k,x \end{pmatrix}$$
$$= f_k(x)\cdot U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{n-2},y_k \end{pmatrix} - f_k(y_k)\cdot U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{n-2},x \end{pmatrix}$$
$$+ \sum_{i=0}^{n-2} (-1)^{n-i} f_k(x_i)\cdot U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{i-1},x_{i+1},\dots,y_k,x \end{pmatrix}.$$

Under the conditions of the theorem, the last term converges to 0 as $k\to\infty$, hence, letting $k\to\infty$ we get

$$0 \leqslant f(x)\cdot U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{n-2},x \end{pmatrix} - \overline{\lim_{k\to\infty}}\, f_k(y_k)\cdot U\begin{pmatrix} 0,\dots,n-1 \\ x_0,\dots,x_{n-2},x \end{pmatrix}$$

whence $f(x) \geqslant \overline{\lim}_{k\to\infty} f_k(y_k)$. We now appeal to Lemma 2.3 for the existence of points $a < t_0 < \cdots < t_{n-3} < x < t_{n-1} < b$ such that

$$U\begin{pmatrix} 0,\dots,n-1 \\ t_0,\dots,t_{n-3},x,t_{n-1} \end{pmatrix} > 0.$$

Proceeding as before we now get

$$0 \leqslant \underline{\lim}_{k\to\infty}\, f_k(y_k)\cdot U\begin{pmatrix} 0,\dots,n-1 \\ t_0,\dots,t_{n-3},x,t_{n-1} \end{pmatrix}$$
$$- f(x)\cdot U\begin{pmatrix} 0,\dots,n-1 \\ t_0,\dots,t_{n-3},x,t_{n-1} \end{pmatrix};$$

that is, $f(x) \leqslant \underline{\lim}_{k\to\infty} f_k(y_k)$. Combining these two inequalities yields $f(x) = \lim_{k\to\infty} f_k(y_k)$, i.e., $\{f_k\}$ converges continuously to $f$ at $x$. At this point we observe that if, in the foregoing, $f_k \equiv f$ for all $k$, then we have shown that $f$ is continuous in $(a, b)$. We may now apply Lemma 2.4 to conclude that $\{f_k\}$ converges uniformly to $f$ on compact subsets of $(a, b)$. ∎

We remark that Theorem 2.1 provides a condition on $U$ which, for a given $x \in (a, b)$, guarantees that each $f \in C(u_0,\dots,u_{n-1})$ is continuous at $x$. We note, further, that Theorem 2.1 holds for T-systems and for non-degenerate WT-systems.

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